An Approach to the Generalization of Lattice Theory - The introduction of "Overpacked Lattices" And Some Results

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June 2025

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Abstract

This paper will discuss and give some results on direct generalizations of mathematical lattices (discrete, periodic subgroups of an n-dimensional space), which we will call "overpacked lattices", and introduce a new method of classifying these lattices.

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1 Introduction

The usual definition of a lattice is a discrete subgroup satisfying periodicity conditions. The most well-known type of lattice in number theory is that of the complex lattice generated by ω_1 *and* ω_2*, or, the fundamental pair of periods. The usual condition is* $\omega_1 \perp_{\mathbb{R}} \omega_2$*, and that the coefficients are in* $\mathbb{Z}$*. We will cover the generalizations of the notion of a lattice both in the complex context and in a general field in this paper.*

2 Results of Research

2.1 Theory

For our generalization of lattices, we can relax the requirement that our basis vectors are independent over $\mathbb{R}$, and instead have our basis vectors over $\mathbb{Z}$ or some such smaller subset, possibly.

Def. 2.1.1. *An overpacked lattice is a lattice over a field, $\mathbb{K}$, which has more generators than it does dimensions of its individual elements, and is required to satisfy $\mathbb{A}$-independence for some $\mathbb{A} \subset \mathbb{Q}$ with $\mathbb{A} = -\mathbb{A}$.*

Def. Ex. 1. *The lattice over $\mathbb{C}$, generated by $\{1, i, \sqrt{2} + i\sqrt{2}\}$ and which is independent over $\mathbb{Z}$ is one such example, as $\sqrt{2} \notin \mathbb{Z}$, and $\mathbb{Z}$ is closed under addition.*

Def. Ex. 2. *A stranger example is the case of the ∞-generated overpacked lattice which is required to have independence over $\mathbb{Z}$, which is defined by the complex numbers $\{\sqrt{2}+i\sqrt{3}, \sqrt{3}+i\sqrt{5}, ..., \}$, where $z_k = \sqrt{p_k} + i\sqrt{p_{k+1}}$. It may seem unintuitive, but this lattice still satisfies $\mathbb{Z}$-independence.*

We can partly extend the theory of the Weierstrass $\wp$-function, Eisenstein series, etc. over lattices to overpacked lattices. The typical definition of the $\wp$-function, for example, is:

$$\wp(z, \Lambda) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

We can easily generalize the $\wp$-function to overpacked lattices via "plugging in" an overpacked lattice, Ω in for Λ, since the definition easily extends.

Thm. 2.1.1. *The $\wp$-function for overpacked lattices is periodic in the generators of the lattice.*

Proof. I: With it being trivial to prove, we know $\Omega + \omega = \Omega$ whenever $\omega \in \Omega$.
II: We know that the sum is taken over all of the lattice points.
Conc: $\wp$ is n-periodic for an overpacked complex lattice generated by n "periods", with the specific set of periods being the generators.

Thm. 2.1.2. *$\wp$ is also an even function over overpacked lattices.*

Proof. This is trivial to prove. See: the proof of evenness for the $\wp$-function in the case of a classical lattice.

In [Har25], we generalized Eisenstein series to a general lattice, interpreting $\Lambda = \mathbb{Z}[i]$ as the original Eisenstein series. Let us recall the definition:

$$E_{2k}^{\Lambda(\omega_1,\omega_2)}(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{1}{(m+n\tau)^{2k}}$$

We can easily generalize this to overpacked lattices, similarly to how we did with $\wp$, by simply plugging in an overpacked lattice.

Def. 2.1.2. *Given an overpacked lattice, Ω, the principal sublattice of type $(b_1, ..., b_n)$, where n is the number of generators of Ω, is defined as $\{S\omega_{k_1} + ... + S\omega_{k_m}\}$ where k_i is the ith projection equal to 1, and m is the number of such projections. Here, $(b_1, ..., b_n) \in \mathbb{Z}_2^n$. We let $W_d(\Omega)$ be the set of principal sublattices such that there are exactly d projections equal to 1. This set has exactly*

$\binom{n}{d}$ *elements.*

Def. 2.1.3. *Similarly to the idea of an "isomorphism" for usual lattices, two overpacked lattices Σ_1 and Σ_2, both with m generators and are summed over the coefficients of a set S, are isomorphic if*

$$\begin{pmatrix} Gen_1(\Sigma_2) \\ . \\ . \\ Gen_n(\Sigma_2) \end{pmatrix} = \begin{pmatrix} a_{11} & . & . & a_{n1} \\ . & & & . \\ . & & & . \\ a_{1n} & . & . & a_{nn} \end{pmatrix} \begin{pmatrix} Gen_1(\Sigma_1) \\ . \\ . \\ Gen_n(\Sigma_1) \end{pmatrix}, |[a_{ij}]| \in \{-1, 1\}.$$

Similarly to the case of "classical" lattices, isomorphism is transitive and reflexive. These properties are easily verifiable.

We immediately obtain the fact that the number of lattice points of a given norm, which we obtain by the theta function for a "classical" lattice, is approximately the sum of the lattice points of that norm of its W_2-sublattices. However, this approximation becomes less and less accurate as the number of generators tends to larger and larger values.

Def. 2.1.4. *Given a lattice, Λ, the determinant of Λ is the determinant of its associated Gram matrix.*

We can just as easily generalize this definition to overpacked lattices, as the definition of a Gram matrix is just a matrix of inner products of a set of vectors (or, in this case, generators).

Def. 2.1.4b. *Given an overpacked lattice, Ω, the determinant of Ω is the determinant of the Gram matrix of its generators.*

2.2 Classification

Def. 2.2.1. *A lattice, Λ, is of type $(S : M_1, ..., M_n :\perp_H)$ if it is defined as a lattice generated as: $\Lambda = \{s_1 \cdot m_1 + ... + s_n \cdot m_n | s_i \in S \forall i\}$, where the generators have to be independent over H, and, $m_i \in M_i \forall i \in \{1...n\}$.*[1]

Immediately, we see it reduces to the notion of a lattice typically considered over a field, $\mathbb{K}$ when dim $m_i = n \forall i$, S=$\mathbb{Z}$, $H = \mathbb{R}$, and, $M =_d \mathbb{K}$. For example, this system indexes a lattice over $\mathbb{C}$ generated by a fundamental pair of periods as $(\mathbb{Z} :_2 \mathbb{C} :\perp_{\mathbb{R}})$.

3 Works Referenced/Utilized

[Har25] Philipp Harland, 2025 - A Beginning Approach to Generalized Modular Forms

[1] We use the notation $(S :_n M :\perp_H)$ if $M_i = M_j = M \forall i, j \in \{1...n\}$. If there are multiple consecutive sets in general, we just sequentially group them together, e.g. $\mathbb{C}, \mathbb{C}, \mathbb{H}, \mathbb{H}, \mathbb{C}$ is represented as ${}_2\mathbb{C},{}_2\,\mathbb{H}, \mathbb{C}$.